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Wavefunctions of a free electron in an external field and their application in intense field interactions: II. Relativistic treatment

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Abstract. The behaviour of a relativistic free electron in an external plane wave field is analysed and a review of the existing solutions of the corresponding Dirac equation is presented. Completeness and orthogonality of the Volkov states are also proved. Based on the exact wavefunction obtained, a relativistic generalisation of the perturbation method proposed in the preceding paper is elaborated as a means of treating intense field problems in a covariant manner.

1. Introduction

In the preceding paper we have presented a review of the solutions of the non-relativistic free electron external field interaction problem (Bergou 1980, to be referred to as I). We have shown the equivalence of some at least superficially different solutions and proposed a perturbation method to treat scattering problems in the presence of an intense external field. In this method we used the complete set of the exact wavefunctions of the free electron in the field as a basis and treated the scattering potential as a perturbation. In the present paper we give a similar account of some existing solutions of the corresponding Dirac equation, prove their equivalence, orthogonality and completeness and, using this complete set of relativistic wavefunctions, we give a simple generalisation of the abovementioned perturbation method and determine the validity of the dipole approximation as well as the validity of the non-relativistic Born approximation in the present problem.

The exact solution of the Dirac equation of a relativistic free electron in an electromagnetic plane wave field has long been known (Volkov 1935). This famous result has, since that time, been reproduced by several authors using different methods. It was shown, for example, that this problem can also be solved by purely algebraic methods (Beers and Nickle 1972). In another paper the so-called projection technique led to the same result (Becker and Mitter 1974). The Dirac equation, however, can also be solved without the direct use of the special assumptions and specific methods applied in these papers. By choosing an appropriate coordinate system, the system of the coupled differential equations for the spinor components can be reduced to an ordinary first-order differential equation for each component separately, if one uses the Majorana representation instead of the standard representation for the Dirac matrices. In this context it is interesting to mention another method (Alperin 1944). It is of course well known that in the 'derivation' of the Dirac equation the originally irrational

Hamiltonian (given by a square root expression) is rationalised by the usual Dirac matrices. The basic idea of Alperin's paper was to exploit the symmetry of the problem by a suitable choice of coordinates, ensuring that both the rational and irrational parts show the required symmetry. Using the wavefunction thus obtained, he determined the scattering cross section of an arbitrarily intense classical EM field by an electron, using the method of the transition currents. The paper did not, however, attract much attention at that time, nor since.

The next section and appendix 1 are devoted to the orthogonality and completeness problem of the Volkov states, this being the central problem in perturbation theoretical applications. In § 3, the solution in the Majorana representation and rederivation of the Alperin solution are given, and their unitary equivalence with the Volkov solution is proved in appendix 2. In § 4, it is shown how the multiphoton radiative corrections to the scattering of a free electron on a background potential due to the interaction with an intense mode of the EM field (laser) can be obtained by using the Volkov states. In the last section we deal with the connection of the present approach with the method introduced by I. The limits of validity of the non-relativistic dipole approximation, as well as other consequences of the relativistic generalisation, are also discussed.

2. The Volkov states

In an external electromagnetic field characterised by the $A(x)$ four-vector potential the relativistic wave equation of a spinor electron has the form†

$$(\mathrm{i}\not{\partial}-\epsilon\not{A}-\kappa)\psi=0 \tag{2.1}$$

where

$$\epsilon=e/\hbar c, \qquad \kappa=mc/\hbar \tag{2.1a}$$

(here c is the velocity of light, and $\hbar$ is Planck's constant divided by 2π). We choose $A(x)$ as representing a transverse plane wave, i.e.

$$A(x)=A(\xi), \qquad \xi=k.x, \qquad k.A=k^2=0. \tag{2.2}$$

In the case of a general elliptically polarised wave

$$\begin{aligned} &A(\xi)=e_1A_1(\xi)+e_2A_2(\xi),\\ &k.e_i=0, \qquad e_i.e_j=-\delta_{ij} \qquad (i,j=1,2). \end{aligned} \tag{2.2a}$$

The well known positive and negative frequency Volkov-type solutions of the above Dirac equation represent modulated plane waves, where the modulation depends only on ξ. The plane wave itself can be parametrised by the four-momentum lying on the free mass shell (initial conditions are not taken into account):

$$\psi=\psi_p^{(\pm)}(x)=\left(1\pm\frac{\epsilon\not{K}\not{A}(\xi)}{2k.p}\right)u_p^{(\pm)}\exp\left\{\mp\mathrm{i}\left[p.x+\int J_p^{(\pm)}(\xi)\,\mathrm{d}\xi\right]\right\}=E_p^{(\pm)}(x)u_p^{(\pm)} \tag{2.3}$$

† The metric tensor $g_{\mu\nu}$ has the components $g_{00}=-g_{ii}=1$ $(i=1,2,3)$ and $g_{\mu\nu}=0$ if $\mu\neq\nu$ $(\mu,\nu=0,1,2,3)$. Space–time coordinates are denoted by x^μ, where $\{x^\mu\}=(ct,\boldsymbol{r})$. Definition of the four-gradient is $\partial=\{\partial_\mu\}$, where $\partial_\mu=\partial/\partial x^\mu$. The scalar product of two four-vectors a and b is $a.b=g_{\mu\nu}a^\mu b^\nu=a_\nu b^\nu=a^0b^0-\boldsymbol{ab}$. Dirac matrices satisfy the anticommutation relations $\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2g_{\mu\nu}$. The $\gamma.a$ type scalar products are denoted by a dagger: $\gamma.a=\not{a}$ (cf Bjorken and Drell (1964); we shall use the metric and notation as well as representation of the Dirac matrices of this reference).

where

$$(\not{p}\mp\kappa)u_p^{(\pm)}=0, \qquad p^2=\kappa^2, \qquad p^\mu=(|p_0|,\boldsymbol{p}) \tag{2.3a}$$

and

$$J_p^{(\pm)}(\xi)=(1/2k\,.\,p)[\pm 2\epsilon p\,.\,A(\xi)-\epsilon^2A^2(\xi)]. \tag{2.3b}$$

One can follow the method of obtaining the solution in this covariant form in a paper by Brown and Kibble (1964). These Volkov states were applied by several authors to treat the interaction of a free electron with an intense optical mode (accounted for by the external field approximation). Interaction with another weak mode or some other weak potential can be taken into account by the usual perturbation theory. As in I, our method is perturbational in the background potential, but not in the photon number n. By this method absorption and emission from the intense mode can be directly computed up to any order in one step, while the usual Feynman–Dyson approach is based on the iterative expansion for the S operator, so higher values of n appear in higher orders of perturbation theory.

For the $E_p^{(\pm)}(x)$ matrices introduced in (2.3) one can easily verify that the following relationships hold (Ritus 1972):

$$\int\frac{d^4x}{(2\pi)^4}\overline{E_p^{(\pm)}(x)}E_q^{(\pm)}(x)=\delta^{(4)}(p-q), \qquad \int\frac{d^4p}{(2\pi)^4}E_p^{(\pm)}(x)\overline{E_p^{(\pm)}(y)}=\delta^{(4)}(x-y),$$

where

$$\bar{E}=\gamma^0E^\dagger\gamma^0.$$

Orthogonality and completeness given in this form are not satisfactory for our purposes since the four-momentum components are not on the free mass shell. Therefore in the following treatment we give different orthogonality and completeness relations. For further investigation of the Volkov states it is convenient to use the light-like components originally introduced by Neville and Rohrlich (1971a, b; see also Becker and Mitter (1974)). This formalism is based on the fact that the vectors

$$n^\mu=\frac{c}{\omega\sqrt{2}}k^\mu=\frac{1}{\sqrt{2}}(1,\boldsymbol{n}), \qquad \hat{n}^\mu=\frac{1}{\sqrt{2}}(1,-\boldsymbol{n}),$$

$$e_i^\mu=(0,\boldsymbol{e}_i) \qquad (i=1,2) \tag{2.4}$$

form a complete orthonormal set in Minkowski space, therefore any 'a' four-vector can be given by its light-like components in the following way:

$$a=na_u+\hat{n}a_v+e_1a_1+e_2a_2 \tag{2.4a}$$

where

$$a_u=\hat{n}\,.\,a, \qquad a_v=n\,.\,a, \qquad a_i=-e_i\,.\,a \qquad (i=1,2). \tag{2.4b}$$

Taking into account (2.4)–(2.4b), the solution (2.3) can be brought to the form

$$\psi_p^{(\pm)}(x)=\left(1\mp\frac{\epsilon}{2p_v}\gamma_v\gamma_ia_i(u)\right)u_p^{(\pm)}\exp\left\{\mp i\left[up_u+vp_v-x_ip_i+\int f_p^{(\pm)}(u)\,du\right]\right\} \tag{2.5}$$

where

$$u=x_v, \qquad v=x_u, \qquad A_i(\xi)=a_i(u) \qquad (i=1,2) \tag{2.5a}$$

and

$$f_p^{(\pm)}(u) = (\epsilon/2p_v)[\mp 2p_i a_i(u) + \epsilon a_i(u)a_i(u)] \tag{2.5b}$$

(there is a summation over repeated indices from $i = 1$ to 2). The states (2.5) (with normalisation factor $(2\pi)^{-3/2}(\kappa/p_v)^{1/2}$) form an orthogonal set in the sense

$$\begin{aligned} &\int \bar{\psi}_{pr}^{(\pm)}(uvx_i)\gamma_v\psi_{p'r'}^{(\pm)}(uvx_i)\,\mathrm{d}v\,\mathrm{d}^2x_i = \delta(p_v - p'_v)\delta^{(2)}(p_i - p'_i)\delta_{rr'}, \\ &\int \bar{\psi}_{pr}^{(\pm)}(uvx_i)\gamma_v\psi_{p'r'}^{(\mp)}(uvx_i)\,\mathrm{d}v\,\mathrm{d}^2x_i = 0, \qquad \bar{\psi} \equiv \psi^{+}\gamma^0. \end{aligned} \tag{2.6}$$

Here $r = 1, 2$ are the spin indices and $\delta^{(2)}$ denotes the two-dimensional Dirac delta function. The normalisation of the $u_p^{(\pm)}$ bispinors is, as usual,

$$\bar{u}_p^{(\pm)}u_p^{(\pm)} = \pm 1. \tag{2.6a}$$

To obtain the appropriate definition of completeness we deal first with the completeness of free plane waves. The solutions of the Dirac equation of a free particle are

$$\varphi_{pr}^{(\pm)}(x) = (2\pi)^{-3/2}(\kappa/p_0)^{1/2}u_{pr}^{(\pm)}\,\mathrm{e}^{\mp \mathrm{i}p.x}. \tag{2.7}$$

The definition and the normalisation condition of the $u_{pr}^{(\pm)}$ bispinors are again given by equations (2.3*a*) and (2.6*a*). The completeness relation of the set of positive and negative frequency solutions is

$$\sum_{r=1,2}\int \mathrm{d}^3p[\varphi_{pr}^{(+)}(x)\bar{\varphi}_{pr}^{(+)}(x') + \varphi_{pr}^{(-)}(x)\bar{\varphi}_{pr}^{(-)}(x')]_{x_0=x'_0}\gamma^0 = \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}'). \tag{2.8}$$

Here we made use of the fact that

$$\sum_{r=1,2} u_{pr}^{(\pm)}\bar{u}_{pr}^{(\pm)} = \frac{\not{p} \pm \kappa}{2\kappa}. \tag{2.8a}$$

Relation (2.8) can be generalised in a covariant manner such that instead of the x_0 = constant three-space we define completeness on a spacelike hyperplane determined by an arbitrary timelike normal vector. For the symmetry of the external plane wave field the best choice is the u = constant null-plane. Therefore, in full analogy with (2.8), to establish completeness of the Volkov states on the null plane we investigate the expression

$$V(x, x') = \sum_{r=1,2}\int \mathrm{d}^3\tilde{p}[\psi_{pr}^{(+)}(x)\bar{\psi}_{pr}^{(+)}(x') + \psi_{pr}^{(-)}(x)\bar{\psi}_{pr}^{(-)}(x')]\gamma_v \tag{2.9}$$

where

$$\int \mathrm{d}^3\tilde{p} = \int_0^\infty \mathrm{d}p_v \iint_{-\infty}^{\infty} \mathrm{d}^2p_i. \tag{2.9a}$$

Before giving the completeness relationship of the Volkov states, we investigate the meaning of the operator defined by (2.9). Using the (2.6) orthogonality relations it is easy to prove the validity of the following projection properties:

$$\int \mathrm{d}^3\tilde{x}'\, V(x, x')\psi_{pr}^{(\pm)}(x') = \psi_{pr}^{(\pm)}(x),$$
$$\int \mathrm{d}^3\tilde{x}'\,\bar{\psi}_{pr}^{(\pm)}(x')\gamma^0 V^{\dagger}(x', x)\gamma^0 = \bar{\psi}_{pr}^{(\pm)}(x), \tag{2.10}$$

where

$$\int \mathrm{d}^3\tilde{x} = \iiint_{-\infty}^{\infty} \mathrm{d}v\, \mathrm{d}^2x_i. \tag{2.10a}$$

Introducing the bra and ket vector notation, the algebraic meaning of the relations (2.10) becomes even more apparent in the abstract state vector space:

$$V(u, u')|\psi_{pr}^{(\pm)}(u')\rangle = |\psi_{pr}^{(\pm)}(u)\rangle, \qquad \langle\psi_{pr}^{(\pm)}(u')|\gamma^0\bar{V}(u', u) = \langle\psi_{pr}^{(\pm)}(u)|\gamma^0, \tag{2.11}$$

where

$$\bar{V}(u', u) = \gamma^0 V^{\dagger}(u', u)\gamma^0. \tag{2.11a}$$

From (2.11) it is clear that $V(u, u')$ and $\bar{V}(u', u)$ represent propagators of the Volkov states and the Dirac-adjoint Volkov states, respectively. It is also clear that $V(u, u)$ and $\bar{V}(u, u)$ are the projectors of the corresponding states. Other abstract algebraic properties of the Volkov states will be discussed in more detail in a subsequent paper.

In the light of the (2.11) projection properties it seems natural to look for the completeness of the Volkov states in the form $[V+\bar{V}]_{u=u'} = \delta^{(3)}(\tilde{x}-\tilde{x}')$. Instead of this relationship, completeness of the Volkov states on a light-like hyperplane can be expressed by the formula

$$[V(x, x') + \gamma^0 V^{\dagger}(x', x)\gamma^0]_{u=u'} = \delta^{(3)}(\tilde{x}-\tilde{x}') - \tfrac{1}{2}\mathrm{i}\kappa\gamma_v\epsilon(v-v')\delta^{(2)}(x_i - x_i'). \tag{2.12}$$

For the derivation of this result and definition of the function $\epsilon(v)$ see appendix 1.

On the basis of (2.12) we have shown in equation (A1.11) that any bispinor function can be represented as a (generalised) linear combination in terms of Volkov states.

3. Connection with other solutions

It is obvious from the preceding section that the problem of a free spinor electron interacting with an external plane wave field can be solved exactly in a covariant manner by using the light-like formalism. In this section we give two important examples where the Hamiltonian form of the corresponding Dirac equation, with appropriate co-ordinate systems, can also be solved exactly.

Let us choose the y axis of our coordinate system as coinciding with the direction of the wavevector of the light field given by the $A(x)$ vector potential, and polarisation parallel to the x axis. The Dirac equation of the problem in this coordinate system is

$$\left[\alpha_x\left(-\mathrm{i}\frac{\partial}{\partial x} - \epsilon A_x\right) + \alpha_y\left(-\mathrm{i}\frac{\partial}{\partial y}\right) + \alpha_z\left(-\mathrm{i}\frac{\partial}{\partial z}\right) + \beta\kappa\right]\psi = \mathrm{i}\frac{\partial}{\partial x_0}\psi \tag{3.1}$$

where

$$A_x = A(\omega\xi/c), \qquad \xi = x_0 - y. \tag{3.1a}$$

A_x is an otherwise arbitrary function of ξ.

We look for the solution of (3.1) again in the form of a plane wave modulated by the external field:

$$\psi = \exp[-\mathrm{i}(x_0 p_0 - x p_x - y p_y - z p_z)]\Phi(\xi). \tag{3.2}$$

Here $\Phi(\xi)$ is a bispinor function for which, after substituting (3.2) into (3.1), we obtain the following ordinary differential equation:

$$\{\alpha_x[p_x - \epsilon A_x(\xi)] + \alpha_y p_y + \alpha_z p_z + \beta\kappa - p_0\}\Phi = (1-\alpha_y)\mathrm{i}\, \mathrm{d}\Phi/\mathrm{d}\xi. \tag{3.3}$$

In the representation of the Dirac matrices used throughout the present paper this is a system of coupled equations, since α_y on the RHS couples the derivatives of the different bispinor components of Φ. It is easy, however, to get rid of this difficulty in the Majorana representation (see appendix 2), where the (3.3) system of equations is decoupled:

$$[\alpha_x(-p_x + \epsilon A_x) + \beta p_y - \alpha_z p_z + \alpha_y \kappa - p_0]\Phi' = (1-\beta)\mathrm{i}\, \mathrm{d}\Phi'/\mathrm{d}\xi. \tag{3.4}$$

Furthermore, it is also shown in appendix 2 that the solution of equation (3.4) has the form

$$\Phi' = \begin{pmatrix} \dfrac{1}{p_\eta}(-p_x\sigma_x - p_z\sigma_z + \kappa\sigma_y)\chi_0' + \dfrac{1}{p_\eta}\epsilon A_x\sigma_x\chi_0' \\ \chi_0' \end{pmatrix} \exp\left(-\mathrm{i}\int J_p^{(+)}(\xi)\,\mathrm{d}\xi\right) \tag{3.5}$$

where χ_0' is a constant spinor, and $J_p^{(+)}$ is defined by (2.3b). The agreement of the phases of the solution $\psi' = \Phi'(\xi)\,\mathrm{e}^{-\mathrm{i}p.x}$ and the transformed Volkov states, which one gets from equation (2.3) after the Majorana transformation, is obvious, while the proof of the equivalence of the bispinor amplitudes is also given in appendix 2.

Another interesting solution of the present problem was given by Alperin (1944). In the following we shall repeat with some modification the original derivation of Alperin's solution. We start from the relativistic energy–momentum formula

$$E/\hbar c = [(\boldsymbol{p}/\hbar - \epsilon\boldsymbol{A})^2 + \kappa^2]^{1/2} \tag{3.6}$$

or, in operator form,

$$\hat{p}_0 = [(\hat{\boldsymbol{p}} - \epsilon\boldsymbol{A})^2 + \kappa^2]^{1/2}, \qquad \hat{p}_0 = \mathrm{i}\,\partial/\partial x_0, \qquad \hat{\boldsymbol{p}} = -\mathrm{i}\,\partial/\partial\boldsymbol{r}. \tag{3.7}$$

In the special coordinate system used throughout in the preceding calculation it is more convenient to take the square root in a different way, namely

$$\hat{p}_x - \epsilon A_x = \mathrm{i}(-\hat{p}_\xi\hat{p}_\eta + \hat{p}_z^2 + \kappa^2)^{1/2} \tag{3.8}$$

where

$$\hat{p}_\xi = \mathrm{i}\left(\frac{\partial}{\partial x_0} - \frac{\partial}{\partial y}\right) = 2\mathrm{i}\frac{\partial}{\partial\xi}, \qquad \hat{p}_\eta = \mathrm{i}\left(\frac{\partial}{\partial x_0} + \frac{\partial}{\partial y}\right) = 2\mathrm{i}\frac{\partial}{\partial\eta},$$
$$\xi = x_0 - y, \qquad \eta = x_0 + y. \tag{3.8a}$$

The matrices $\alpha_\xi = \frac{1}{2}\mathrm{i}(\alpha_y + \mathrm{i}\alpha_x)$ and $\alpha_\eta = \frac{1}{2}\mathrm{i}(\alpha_y - \mathrm{i}\alpha_x)$ satisfy the commutation relation

$$\alpha_\xi\alpha_\eta + \alpha_\eta\alpha_\xi = -1. \tag{3.9}$$

By using these matrices the Dirac equation corresponding to (3.8) has the following rational form:

$$(\hat{p}_x - \epsilon A_x)\psi = \mathrm{i}(\alpha_\xi\hat{p}_\xi + \alpha_\eta\hat{p}_\eta + \alpha_z\hat{p}_z + \beta\kappa)\psi. \tag{3.10}$$

The solution can be looked for again with the usual ansatz

$$\psi = \exp[\mathrm{i}(xp_x + zp_z - \tfrac{1}{2}\xi p_\xi - \tfrac{1}{2}\eta p_\eta)]\phi(\xi), \qquad p_\xi = p_0 + p_y, \qquad p_\eta = p_0 - p_y. \tag{3.11}$$

After substituting (3.11) into (3.10) we obtain a coupled system of equations for the components of ϕ:

$$-\mathrm{i}[p_x - \epsilon A_x(\xi)]\phi_1 = -p_\eta\phi_4 + p_z\phi_3 + \kappa\phi_1, \tag{3.12a}$$

$$-\mathrm{i}[p_x - \epsilon A_x(\xi)]\phi_2 = p_\xi\phi_3 + 2\mathrm{i}\, \mathrm{d}\phi_3/\mathrm{d}\xi - p_z\phi_4 + \kappa\phi_2, \tag{3.12b}$$

$$-\mathrm{i}[p_x - \epsilon A_x(\xi)]\phi_3 = -p_\eta\phi_2 + p_z\phi_1 - \kappa\phi_3, \tag{3.12c}$$

$$-\mathrm{i}[p_x - \epsilon A_x(\xi)]\phi_4 = p_\xi\phi_1 + 2\mathrm{i}\, \mathrm{d}\phi_1/\mathrm{d}\xi - p_z\phi_2 - \kappa\phi_4. \tag{3.12d}$$

From (3.12*a*) and (3.12*c*) ϕ_2 and ϕ_4 can be expressed by ϕ_1 and ϕ_3:

$$\phi_2 = (1/p_\eta)\{\mathrm{i}[(p_x - \epsilon A_x) + \mathrm{i}\kappa]\phi_3 + p_z\phi_1\}, \tag{3.13a}$$

$$\phi_4 = (1/p_\eta)\{\mathrm{i}[(p_x - \epsilon A_x) - \mathrm{i}\kappa]\phi_1 + p_z\phi_3\}, \tag{3.13b}$$

and substituting these expressions into (3.12*b*) and (3.12*d*) we obtain two similar uncoupled equations for ϕ_1 and ϕ_3:

$$2\mathrm{i}\frac{\mathrm{d}\phi_{1,3}}{\mathrm{d}\xi} = \left\{\frac{1}{p_\eta}[(p_x - \epsilon A_x)^2 + p_z^2 + \kappa^2] - p_\xi\right\}\phi_{1,3}. \tag{3.14}$$

The solution of (3.14), taking into account (A2.6*b*), will be

$$\phi_{1,3} = \phi_{1,3}(0)\exp\left(-\mathrm{i}\int J_p^{(+)}(\xi)\,\mathrm{d}\xi\right), \qquad \phi_{1,3}(0) = \text{constant}. \tag{3.14a}$$

Through (3.13*a*)–(3.14*a*), all four components of ϕ are known, and thus another solution of the Dirac equation is found. The function in the exponent of this wavefunction coincides with exponents of the Volkov states and the state found in the Majorana representation. All we have to show is the equivalence of the bispinor part with the previous solutions. From (A2.5)

$$-\mathrm{i}\varphi_1' = (1/p_\eta)\{[\mathrm{i}(p_x - \epsilon A_x) - \kappa]\chi_2' + p_z\mathrm{i}\chi_1'\}, \tag{3.15a}$$

$$\varphi_2' = (1/p_\eta)\{[\mathrm{i}(p_x - \epsilon A_x) + \kappa]\mathrm{i}\chi_1' + p_z\chi_2'\}. \tag{3.15b}$$

Comparing the above relations with (3.13*a*) and (3.13*b*), we can immediately see that the same relation holds between $\binom{-\mathrm{i}\varphi_1'}{\varphi_2'}$ and $\binom{\mathrm{i}\chi_1'}{\chi_2'}$ as between $\binom{\phi_2}{\phi_4}$ and $\binom{\phi_1}{\phi_3}$; therefore if we make the identifications $\mathrm{i}\chi_1' \to \phi_1$ and $\chi_2' \to \phi_3$ the corresponding $-\mathrm{i}\varphi_1' \to \phi_2$ and $\varphi_2' \to \phi_4$ identification must also hold. From this consideration the connection between ϕ and $\binom{\varphi'}{\chi'}$ can be written in the following compact form:

$$\phi = \begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix} = \begin{pmatrix}0 & 0 & \mathrm{i} & 0\\ -\mathrm{i} & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\end{pmatrix}\begin{pmatrix}\varphi_1'\\ \varphi_2'\\ \chi_1'\\ \chi_2'\end{pmatrix} = T\begin{pmatrix}\varphi'\\ \chi'\end{pmatrix}. \tag{3.16}$$

As the T matrix defined here is unitary, the Alperin-type solutions are equivalent with the solutions obtained in the Majorana representation, and due to the transitivity property of the unitary transformations the three formally different wavefunctions considered so far are interrelated by unitary transformations and they are therefore equivalent from the physical point of view.

4. An application of the Volkov states

In the preceding paper the wavefunctions of a non-relativistic free electron moving in a homogeneous external field (dipole approximation) were used as the basis set of a perturbation method to calculate the cross section of the inverse as well as induced multiphoton bremsstrahlung process (Bergou 1980). In this section we work out an obvious generalisation of the method for the relativistic case and beyond dipole approximation by using the Volkov states as the basis set. Similar problems were touched on earlier (Denisov and Fedorov 1967, Brehme 1971) where the analytical and numerical behaviour of the relativistic cross section formulae of the scattering by a Coulomb background were investigated by different methods, and in a recent paper (Ehlotzky 1978) results beyond dipole approximation but using non-relativistic description were published.

Consider the problem of the scattering of a relativistic free electron by a $V(\boldsymbol{r})$ scalar background potential in the presence of an intense electromagnetic mode (laser light). The intense mode can be accounted for by the external field approximation and the corresponding Dirac equation reads (using light-like formalism)

$$\{\gamma_v \mathrm{i}\,\partial_v + \gamma_u \mathrm{i}\,\partial_u - \gamma_i[\mathrm{i}\,\partial_i - \epsilon a_i(u)] - \epsilon \not{V}(v-u, x_i) - \kappa\}\psi = 0 \tag{4.1}$$

where

$$\partial_u = \partial/\partial u, \qquad \partial_v = \partial/\partial v, \qquad \partial_i = \partial/\partial x^i \qquad (i=1,2) \tag{4.1a}$$

and

$$\not{V}(v-u, x_i) = \gamma^0 V(\boldsymbol{r}) \tag{4.1b}$$

is the background potential.

The purpose of this section is to determine the transition amplitude and scattering cross section of the process $\psi_{qr}^{(+)} \to \psi_{q'r'}^{(+)}$ caused by the scattering potential $V(\boldsymbol{r})$. To this end we express the solution of equation (4.1) with the help of usual perturbation theory as a series in powers of $V(\boldsymbol{r})$, and we shall restrict ourselves to the linear term in $V(\boldsymbol{r})$. That is, we use the Born approximation, and the validity of the method is reduced to the question of the validity of this approximation.

We look for the solution in the form

$$\psi = \psi_{qr}^{(+)} + \psi_k \tag{4.2}$$

where q, r are determined by the parameters of the initial state and the correction term ψ_k is a superposition of the (2.5) Volkov states taking into account the (A1.11) expansion theorem:

$$\psi_k(uvx_i) = \sum_{r=1,2} \int \mathrm{d}^3\tilde{p}[b_{pr}^{(+)}(u)\psi_{pr}^{(+)}(uvx_i) + b_{pr}^{(-)}(u)\psi_{pr}^{(-)}(uvx_i)]. \tag{4.3}$$

Upon substitution of (4.3) into (4.1) we obtain the equation

$$\sum_{r=1,2} \int \mathrm{d}^3\tilde{p}\, \gamma_v \mathrm{i}\left(\frac{\mathrm{d}b_{pr}^{(+)}}{\mathrm{d}u}\psi_{pr}^{(+)} + \frac{\mathrm{d}b_{pr}^{(-)}}{\mathrm{d}u}\psi_{pr}^{(-)}\right) = \epsilon \not{V}\psi_{qr}^{(+)} + \epsilon \not{V}\psi_k. \tag{4.3a}$$

According to (A1.12)

$$\gamma_v b_{pr}^{(\pm)} = \gamma_v c_{pr}^{(\pm)} \tag{4.3b}$$

since $\gamma_v^2 = 0$. Taking into account this relationship equation (4.3*a*) can be further simplified to yield

$$\sum_{r=1,2} \int \mathrm{d}^3\tilde{p}\, \gamma_v \mathrm{i}\left(\frac{\mathrm{d}c_{pr}^{(+)}}{\mathrm{d}u}\psi_{pr}^{(+)} + \frac{\mathrm{d}c_{pr}^{(-)}}{\mathrm{d}u}\psi_{pr}^{(-)}\right) = \epsilon \not{V}\psi_{qr}^{(+)} + \epsilon\not{V}\psi_k. \tag{4.4}$$

Here $c_{pr}^{(+)}$ and $c_{pr}^{(-)}$ are scalar amplitudes to be determined. For the sake of simplicity we choose the initial conditions

$$c_{pr}^{(\pm)}(u \leqslant u_i) = 0 \qquad \text{for all } p \text{ and } r. \tag{4.4a}$$

In the spirit of the Born approximation we drop the second term on the RHS of equation (4.4) which contains the product of the correction term and the perturbing potential, thus giving higher-order corrections only. One should note, though, that it is not necessarily true that the neglected corrections are smaller than the term $\epsilon\not{V}\psi_{qr}^{(+)}$ which we keep. Thus, conditions for the validity of our method are the same as for the Born approximation. Then we take the scalar product of the remaining terms with $\bar{\psi}_{q'r'}^{(+)}$ from the left and obtain the following ordinary differential equation for $c_{pr}^{(+)}(u, u_i)$:

$$\mathrm{i}\, \mathrm{d}c_{q'r'}^{(+)}/\mathrm{d}u = \int \mathrm{d}v\, \mathrm{d}^2x_i\, \bar{\psi}_{q'r'}^{(+)}(uvx_i)\epsilon\not{V}\psi_{qr}^{(+)}(uvx_i). \tag{4.5}$$

Here we have directly made use of the (2.6) orthogonality relations and that for arbitrary spin orientation $\bar{u}_p^{(+)}\gamma_v u_p^{(+)} = (p_v/\kappa)\bar{u}_p^{(+)}u_p^{(+)}$ and $\bar{u}_{-p}^{(+)}\gamma_v u_p^{(-)} = 0$. Equation (4.5) can be integrated in a simple way, leading to

$$c_{q'r'}^{(+)}(u, u_i) = -\mathrm{i}\epsilon \int_{u_i}^{u} \mathrm{d}u' \int \mathrm{d}v\, \mathrm{d}^2x_i\, \bar{\psi}_{q'r'}^{(+)}\not{V}\psi_{qr}^{(+)}. \tag{4.5a}$$

The transition matrix element of the $qr \to q'r'$ process is connected with the $c_{q'r'}^{(\pm)}(u, u_i)$ amplitude in the following way:

$$T_{fi} = c_{q'r'}^{(+)}(u \to \infty, u_i \to -\infty) \tag{4.6}$$

or

$$T_{fi} = -\mathrm{i}\epsilon \int \mathrm{d}^4x\, \bar{\psi}_{q'r'}^{(+)}\not{V}\psi_{qr}^{(+)}. \tag{4.6a}$$

We perform the calculation for a circularly polarised wave

$$A_1(\xi) = a\cos\xi, \qquad A_2(\xi) = a\sin\xi. \tag{4.7}$$

Then from (4.6)

$$T_{fi} = -\mathrm{i}\epsilon \int \mathrm{d}^4x\, \bar{u}_{q'r'}^{(+)}\left(1 - \frac{\epsilon k\not{A}}{2q'\,.\,k}\right)\gamma^0\left(1 + \frac{\epsilon k\not{A}}{2q\,.\,k}\right)u_{qr}^{(+)}$$
$$\times V(\boldsymbol{r})\exp\{\mathrm{i}[(q_a' - q_a)\,.\,x + z\sin(k\,.\,x - \chi)]\}, \tag{4.8}$$

$$z = (\alpha_1^2 + \alpha_2^2)^{1/2}, \qquad \alpha_{1,2} = \epsilon a\left(\frac{q'\,.\,e_{1,2}}{2q'\,.\,k} - \frac{q\,.\,e_{1,2}}{2q\,.\,k}\right),$$
$$\sin\chi = \alpha_2/z, \qquad q_a = q + (\epsilon^2a^2/2k\,.\,q)k. \tag{4.8a}$$

To evaluate the integral in (4.8) we use the well known Fourier expansion

$$e^{iz\sin\varphi}=\sum_{n=-\infty}^{\infty} J_n(z)\,e^{in\varphi}$$

where J_n denotes a Bessel function of integer order. Thus the transition matrix element represents an infinite sum of photon absorbed and photon emitted terms:

$$T_{fi}=\sum_{n=-\infty}^{\infty} T_{fi}^{(n)},\qquad T_{fi}^{(n)}=-2\pi i t_{fi}^{(n)}\delta(q_a^{0\prime}-q_a^0+nk^0), \tag{4.9}$$

$$t_{fi}^{(n)}=\epsilon V(\boldsymbol{q}_n)(\bar{u}_{q'r'}^{(+)}M_n u_{qr}^{(+)}),$$
$$\boldsymbol{Q}_n=\boldsymbol{q}'_a-\boldsymbol{q}_a+n\boldsymbol{k},\qquad V(\boldsymbol{Q}_n)=\int d^3\boldsymbol{r}\,V(\boldsymbol{r})\,e^{-i\boldsymbol{Q}_n\boldsymbol{r}}. \tag{4.9a}$$

Relation (4.9) expresses, in an explicit way, energy conservation; the index a stands for the fact that electron energies in the presence of the external field are different from those of a bare electron (see e.g. the last of the (4.8a) relations). In (4.9a)

$$M_n=\left(\gamma_0+\frac{\epsilon^2a^2}{4q_vq'_v}\gamma_v\right)C_n+\left(\frac{\epsilon a}{2q_v}\gamma_0\gamma_v+\frac{\epsilon a}{2q'_v}\gamma_v\gamma_0\right)(\not{e}^{(+)}C_{n-1}+\not{e}^{(-)}C_{n+1}) \tag{4.10}$$

where

$$C_n=J_n(z)\,e^{-inx},\qquad e^{(+)}=\tfrac{1}{2}(e_1-ie_2)=e^{(-)*}. \tag{4.10a}$$

After averaging over initial and summing up for final spin variables one obtains finally for the scattering cross section

$$\frac{d\sigma^{(n)}}{d\Omega}=\frac{|\boldsymbol{q}'|}{|\boldsymbol{q}|}J_n^2(z)\frac{d\sigma_B^{(n)}}{d\Omega}\frac{1}{2}\left(1+\frac{q_0q'_0+\boldsymbol{q}\boldsymbol{q}'}{\kappa^2}\right)\left(\frac{q'_0}{\kappa}\right)$$
$$+\frac{|\boldsymbol{q}'|}{|\boldsymbol{q}|}\frac{d\sigma_B^{(n)}}{d\Omega}\left(\frac{q'_0}{\kappa}\right)(\alpha_n\nu+\beta_n\nu^2+\gamma_n\nu^3+\delta_n\nu^4) \tag{4.11}$$

where

$$\frac{d\sigma_B^{(n)}}{d\Omega}=\left|\frac{\kappa}{2\pi}\epsilon V(\boldsymbol{Q}_n)\right|^2,\qquad \nu=\frac{\epsilon a}{\kappa}(\nu^2\approx 8\cdot 10^{-11}\lambda^2 I). \tag{4.11a}$$

Here I is the intensity of light in W cm^{-2} units, λ is the wavelength in centimetres and ν is the dimensionless intensity parameter used by several authors.

The parameters α_n, β_n, γ_n and δ_n depend upon the four-momentum of the electron as well as on the frequency and polarisation of the light field and they are bounded functions of light intensity. Their analytical expression is rather complicated and does not give a better insight into the physical process involved, and therefore we omit them here. Equation (4.11) together with equation (4.11a) represents the main result of this section. In (4.11) the first term is just the generalisation of the result obtained by the non-relativistic dipole approximation for the nonlinear direct and inverse bremsstrahlung, while the second term comes from the interaction of the spin momentum with the EM field and is exact in the sense that in all orders it is given by a fourth-order polynomial of the intensity parameter ν, only the coefficients being slowly dependent on the order of the process and intensity.

5. Discussion and summary

As is well known, an intense mode of the electromagnetic field can be represented by a c-number plane wave field. The central problem of the semiclassical theory is, therefore, the solution of the wave equations of charged particles in such a surrounding. As an extension of previous work (Bergou 1980) on exact wavefunctions, in the present paper we have given a detailed study of the Volkov states from some special aspects. In § 2, using the light-like formalism we have shown that the Volkov states parametrised by the four-momentum on the free mass-shell form a complete orthonormal set on the $k \, . \, x = \text{constant}$ null-plane. The orthogonality and completeness of this kind are consequences of the special symmetry of the external plane wave field, i.e. of the dependence of the vector potential on the quantity $k \, . \, x$ only. As several authors have made direct use of these solutions in perturbation theoretical calculations of different kinds, it seemed to us to be important to prove the completeness of this system and to examine in what sense they can be applied as a basis set.

In the next section we gave two simple methods for the solutions of the Dirac equation under consideration. Each of the methods was based on the fact that, with appropriate choice of the coordinate system, the coupled system of equations for the bispinor components can be decoupled into ordinary differential equations for each component separately in a suitable representation for the Dirac matrices. This was first performed in the Majorana representation and another solution was found by a suitable rationalisation of the relativistic energy–momentum formula. We note here that neither of these two methods of solution required the solution of a second-order equation as was done in the original derivation by Volkov. We have shown that the bispinor amplitudes of the solutions found in this way are related to the Volkov amplitudes through unitary transformations (the agreement of phases is obvious) and consequently they are equivalent to each other from the physical point of view.

In the last section the use of the Volkov states was demonstrated in the derivation of the nonlinear inverse and induced bremsstrahlung scattering cross section. The expression obtained can be considered as a relativistic generalisation of the results obtained in the non-relativistic dipole approximation. Scattering is elastic with respect to the background potential and inelastic with respect to the external field. This last property is expressed by the Bessel functions, while corrections to this result were found from two different origins. The first is what one would expect when the non-relativistic dipole approximation is dropped (relativistic non-dipole part) and the second comes from the relativistic interaction of a spin momentum with an external field. It is interesting to note at this point that this second correction is given by a similar finite (fourth-order) polynomial of the intensity parameter in all orders, the coefficients of the polynomial being only slowly dependent on the order. From here we may conclude that in a sufficiently intense external field, relativistic effects may become important.

Appendix 1

In order to prove equation (2.12), we start with the definition (2.9) of the Volkov propagator. We attach the normalisation constants $(2\pi)^{-3/2}(\kappa/p_v)^{1/2}$ to the Volkov states (2.3) and express the particular value $V(x, x')|_{u=u'}$ on the $u = \text{constant}$ hyperplane as

$$V(x,x')|_{u=u'}=\int\frac{\mathrm{d}^3\tilde{p}}{(2\pi)^3}\frac{\kappa}{p_v}\bigg[\left(1+\frac{\epsilon \not{k}\not{A}(\xi)}{2k\,.\,p}\right)\sum_{r=1,2}u^{(+)}_{pr}\bar{u}^{(+)}_{pr}\gamma^0\left(1+\frac{\epsilon \not{k}\not{A}(\xi)}{2k\,.\,p}\right)^{\dagger}\gamma^0\gamma_v\,\mathrm{e}^{-\mathrm{i}(\tilde{x}-\tilde{x})\tilde{p}}$$
$$+\left(1-\frac{\epsilon \not{k}\not{A}(\xi)}{2k\,.\,p}\right)\sum_{r=1,2}u^{(-)}_{pr}\bar{u}^{(-)}_{pr}\gamma^0\left(1-\frac{\epsilon \not{k}\not{A}(\xi)}{2k\,.\,p}\right)^{\dagger}\gamma^0\gamma_v\,\mathrm{e}^{\mathrm{i}(\tilde{x}-\tilde{x}')\tilde{p})}\bigg] \qquad \text{(A1.1)}$$

where $(\tilde{x}-\tilde{x}')\tilde{p}=(v-v')p_v-(x_i-x'_i)p_i$.

In the representation used throughout this paper $\gamma^0\gamma^{\dagger}_{\mu}\gamma^0=\gamma_\mu$, and by using the transversality condition $k\,.\,A=0$ we have $\gamma^0(\not{k}\not{A})^{\dagger}\gamma^0=\not{A}\not{k}$. Using this and the addition theorem (2.8a) for the dyadic sum of the free bispinors, and changing the integration variable from $\tilde{p}$ to $-\tilde{p}$ in the negative frequency term of (A1.1), we obtain for $V(x,x')|_{u=u'}$

$$V(x,x')|_{u=u'}=\int_{-\infty}^{\infty}\frac{\mathrm{d}p_v}{2\pi}\iint_{-\infty}^{\infty}\frac{\mathrm{d}^2p_i}{(2\pi)^2}\left(1+\frac{\epsilon \not{k}\not{A}(\xi)}{2k\,.\,p}\right)\frac{\not{p}+\kappa}{2p_v}\left(1-\frac{\epsilon \not{k}\not{A}(\xi)}{2k\,.\,p}\right)\gamma_v\,\mathrm{e}^{-\mathrm{i}(\tilde{x}-\tilde{x}')\tilde{p}}. \qquad \text{(A1.2)}$$

Similarly, for the Dirac adjoint propagator, defined by equation (2.11a), we have

$$\bar{V}(x,x')|_{u=u'}=\overline{V(x',x)|_{u=u'}}$$
$$=\int_{-\infty}^{\infty}\frac{\mathrm{d}p_v}{2\pi}\iint_{-\infty}^{\infty}\frac{\mathrm{d}^2p_i}{(2\pi)^2}\gamma_v\left(1+\frac{\epsilon \not{k}\not{A}(\xi)}{2k\,.\,p}\right)\frac{\not{p}+\kappa}{2p_v}\left(1-\frac{\epsilon \not{k}\not{A}(\xi)}{2k\,.\,p}\right)\mathrm{e}^{-\mathrm{i}(\tilde{x}-\tilde{x}')\tilde{p}}. \qquad \text{(A1.3)}$$

Taking into account $\not{p}\gamma_v+\gamma_v\not{p}=2p_v$, we obtain the expression for the sum of (A1.2) and (A1.3),

$$[V(x,x')=\bar{V}(x,x')]_{u=u'}=\delta^{(3)}(\tilde{x}-\tilde{x}')-\tfrac{1}{2}\mathrm{i}\kappa\gamma_v\epsilon(v-v')\delta^{(2)}(x_i-x'_i) \qquad \text{(A1.4)}$$

where

$$\epsilon(v)=\frac{\mathrm{i}}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}vp_v}}{p_v}\,\mathrm{d}p_v \qquad \text{(A1.5)}$$

(see also Neville and Rohrlich 1971a, appendix II).

By (A1.4) and the definition of $V(x,x')$ an arbitrary bispinor ψ can now be expanded in terms of the Volkov states in the following way:

$$\psi(uvx_i)=\sum_{r=1,2}\int\mathrm{d}^3\tilde{p}\{[c^{(+)}_{pr}(u)+\gamma_v d^{(+)}_{pr}(u)]\psi^{(+)}_{pr}(uvx_i)$$
$$+[c^{(-)}_{pr}(u)+\gamma_v d^{(-)}_{pr}(u)]\psi^{(-)}_{pr}(uvx_i)\}+\tfrac{1}{2}\mathrm{i}\kappa\gamma_v\int_{-\infty}^{\infty}\mathrm{d}v'\,\epsilon(v-v')\psi(uv'x_i) \qquad \text{(A1.6)}$$

where

$$c^{(\pm)}_{pr}(u)=\int\mathrm{d}^3\tilde{x}\,\bar{\psi}^{(\pm)}_{pr}(uvx_i)\gamma_v\psi(uvx_i) \qquad \text{(A1.7)}$$

and

$$d^{(\pm)}_{pr}(u)=\int\mathrm{d}^3\tilde{x}\,\bar{\psi}^{(\pm)}_{pr}(uvx_i)\psi(uvx_i). \qquad \text{(A1.8)}$$

If in the last term of the RHS in (A1.6) $\psi(uv'x_i)$ is regarded as being expressed from (A1.6), then $\psi(uvx_i)$ takes the form

$$\psi(uvx_i)=\sum_{r=1,2}\int \mathrm{d}^3\tilde{p}\Big[[c_{pr}^{(+)}(u)+\gamma_v d_{pr}^{(+)}(u)]\psi_{pr}^{(+)}(uvx_i)+[c_{pr}^{(-)}(u)+\gamma_v d_{pr}^{(-)}(u)]\psi_{pr}^{(-)}(uvx_i)$$
$$+\tfrac{1}{2}\mathrm{i}\kappa\gamma_v\Big(c_{pr}^{(+)}(u)\int_{-\infty}^{\infty}\epsilon(v-v')\psi_{pr}^{(+)}(uv'x_i)\,\mathrm{d}v'$$
$$+c_{pr}^{(-)}(u)\int_{-\infty}^{\infty}\epsilon(v-v')\psi_{pr}^{(-)}(uv'x_i)\,\mathrm{d}v'\Big)\Big]. \tag{A1.9}$$

Here we made use of the fact that γ_v is nilpotent, or else $\gamma_v^2=0$. The Volkov states $\psi_{pr}^{(\pm)}(uv'x_i)$ depend on v' only through the exponential factors $\mathrm{e}^{\mp\mathrm{i}v'p_v}$. Hence, the integrals in equation (A1.9) involving $\epsilon(v-v')$ can be simply evaluated, if we take into account equation (A1.5), giving

$$\int_{-\infty}^{\infty}\mathrm{d}v'\,\epsilon(v-v')\psi_{pr}^{(\pm)}(uv'x_i)=\pm\mathrm{i}(2/p_v)\psi_{pr}^{(\pm)}(uvx_i). \tag{A1.10}$$

Introducing this last expression into (A1.9), we obtain

$$\psi(uvx_i)=\sum_{r=1,2}\int \mathrm{d}^3\tilde{p}[b_{pr}^{(+)}(u)\psi_{pr}^{(+)}(uvx_i)+b_{pr}^{(-)}(u)\psi_{pr}^{(-)}(uvx_i)] \tag{A1.11}$$

where

$$b_{pr}^{(\pm)}(u)=c_{pr}^{(\pm)}(u)+\gamma_v d_{pr}^{(\pm)}(u)\mp\kappa\gamma_v(c_{pr}^{(\pm)}(u)/p_v). \tag{A1.12}$$

Expression (A1.11) can be regarded as the desired expansion in terms of the Volkov states and it is fully satisfactory for the purposes of any practical application (cf equation (4.3)).

Appendix 2

In the standard representation used throughout the present paper, the Dirac matrices $\alpha_{x,y,z}$ and β have the form

$$\alpha_{x,y,z}=\begin{pmatrix}0 & \sigma_{x,y,z}\\ \sigma_{x,y,z} & 0\end{pmatrix},\qquad \beta=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{A2.1}$$

where $\sigma_{x,y,z}$ are 2×2 Pauli matrices and 0 and 1 are 2×2 zero and unity matrices. The corresponding γ matrices are $\gamma^{1,2,3}=\gamma_{x,y,z}=\beta\alpha_{x,y,z}$, $\gamma^0=\beta$. In equation (3.3) of § 3 the matrix $(1-\alpha_y)$ on the RHS has non-diagonal elements as well, and the component derivatives of Φ are coupled. For this reason, it is convenient to transform equation (3.3) to a form where the matrix coefficient on the RHS is diagonal. Therefore we introduce the self-adjoint unitary matrix (Majorana 1937)

$$u_{\mathrm{M}}=(1/\sqrt{2})(\alpha_y+\beta),\qquad u_{\mathrm{M}}^{-1}=u_{\mathrm{M}}^{+}=u_{\mathrm{M}}. \tag{A2.2}$$

One can easily prove the validity of the relations

$$\begin{aligned}\alpha'_x&=u_{\mathrm{M}}\alpha_x u_{\mathrm{M}}^{-1}=-\alpha_x, & \alpha'_y&=u_{\mathrm{M}}\alpha_y u_{\mathrm{M}}^{-1}=\beta,\\ \alpha'_z&=u_{\mathrm{M}}\alpha_z u_{\mathrm{M}}^{-1}=-\alpha_z, & \beta'&=u_{\mathrm{M}}\beta u_{\mathrm{M}}^{-1}=\alpha_y.\end{aligned} \tag{A2.3}$$

With the aid of these relations the transformed equation corresponding to equation (3.3) reads in the Majorana representation as

$$[\alpha_x(-p_x+\epsilon A_x)+\beta p_y-\alpha_z p_z+\alpha_y\kappa-p_0]\Phi'=(1-\beta)\mathrm{i}\,\mathrm{d}\Phi'/\mathrm{d}\xi \tag{3.4}$$

where

$$\Phi'=u_{\mathrm{M}}\Phi. \tag{A2.4}$$

If we introduce φ' and χ', the upper and lower components of Φ' respectively, (3.4) gives a simple algebraic relation between φ' and χ':

$$\varphi'=(p_0-p_y)^{-1}[(-p_x+\epsilon A_x)\sigma_x-p_z\sigma_z+\kappa\sigma_y]\chi'. \tag{A2.5}$$

If (A2.5) is substituted into the lower component equation of (3.4) we obtain

$$2\mathrm{i}\,\mathrm{d}\chi'/\mathrm{d}\xi=\{(1/p_\eta)[(p_x-\epsilon A_x)^2+p_z^2+\kappa^2]-p_\xi\}\chi' \tag{A2.6}$$

where

$$p_\eta=p_0-p_y, \qquad p_\xi=p_0+p_y. \tag{A2.6a}$$

Without loss of generality we can assume that the parameter p satisfies the usual free mass-shell relationship

$$p_0^2=p_x^2+p_y^2+p_z^2+\kappa, \qquad p_\eta p_\xi=p_x^2+p_z^2+\kappa^2. \tag{A2.6b}$$

Then from (A2.6)

$$\begin{aligned}&\chi'=\exp\left(\mathrm{i}\int I_p(\xi)\,\mathrm{d}\xi\right)\chi_0',\\ &I_p(\xi)=(1/2p_\eta)[2\epsilon A_x(\xi)p_x-\epsilon^2A_x^2(\xi)]=-J_p^{(+)}(\xi).\end{aligned} \tag{A2.7}$$

Here χ_0' is a constant spinor. The final form of the solution of (3.4) using (A2.5) then becomes

$$\Phi'=\begin{pmatrix}\dfrac{1}{p_\eta}(-p_x\sigma_x-p_z\sigma_z+\kappa\sigma_y)\chi_0'+\dfrac{1}{p_\eta}\epsilon A_x\sigma_x\chi_0'\\ \chi_0'\end{pmatrix}\exp\left(-\mathrm{i}\int J_p^{(+)}(\xi)\,\mathrm{d}\xi\right). \tag{3.5}$$

The exponent of (3.5) agrees with that of the Volkov states, while the proof of the equivalence of the bispinor amplitudes will be given in what follows. In the special coordinate system introduced in the above calculation the positive-frequency Volkov bispinor reads

$$u_p^{(v)}=(1+\epsilon \not{K}\not{A}/2k\,.\,p)u_p=[1-(1+\alpha_y)\alpha_x(\epsilon A_x/2p_\eta)]u_p. \tag{A2.8}$$

In the Majorana representation this becomes

$$u_p^{(v)\prime}=[1+(1+\beta)\alpha_x(\epsilon A_x/2p_\eta)]u_p' \tag{A2.9}$$

where now u_p' satisfies the transformed free energy eigenvalue-equation

$$p_0u_p'=(\boldsymbol{\alpha}'\boldsymbol{p}+\beta'\kappa)u_p'=(-\alpha_xp_x+\beta p_y-\alpha_zp_z+\alpha_y\kappa)u_p'. \tag{A2.9a}$$

Equation (A2.9) written out explicitly reads now

$$\left[1+\begin{pmatrix}0&\sigma_x\\0&0\end{pmatrix}\frac{\epsilon A_x}{p_\eta}\right]\begin{pmatrix}\varphi_0'\\ \chi_0'\end{pmatrix}=\begin{pmatrix}\varphi_0'+(\epsilon A_x/p_\eta)\sigma_x\chi_0'\\ \chi_0'\end{pmatrix}. \tag{A2.10}$$

On the other hand, from (A2.9a)

$$\varphi_0' = (1/p_\eta)(-p_x\sigma_x - p_z\sigma_z + \kappa\sigma_y)\chi_0'. \tag{A2.11}$$

Substituting (A2.11) into (A2.10) and comparing the result with (3.5), one can see that the solution given here is equivalent to the Volkov solution.

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